Invariant classification and the generalised invariant formalism: conformally flat pure radiation metrics, with zero cosmological constant.*

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Abstract.

Metrics obtained by integrating within the generalised invariant formalism are structured around their intrinsic coordinates, and this considerably simplifies their invariant classification and symmetry analysis. We illustrate this by presenting a simple and transparent complete invariant classification of the conformally flat pure radiation metrics (except plane waves) in such intrinsic coordinates; in particular we confirm that the three apparently non-redundant functions of one variable are genuinely non-redundant, and easily identify the subclasses which admit a Killing and/or a homothetic Killing vector. Most of our results agree with the earlier classification carried out by Skea in the

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*This is an expanded version of the publication [3]. Also some typos and numerical coefficients have been corrected compared to the published version.

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different Koutras-McIntosh coordinates, which required much more involved calculations; but there are some subtle differences. Therefore, we also rework the classification in the Koutras-McIntosh coordinates, and by paying attention to some of the subtleties involving arbitrary functions, we are able to obtain complete agreement with the results obtained in intrinsic coordinates. In particular, we have corrected and completed statements and results by Edgar and Vickers, and by Skea, about the orders of Cartan invariants at which particular information becomes available.

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1 Introduction

1.1 Integration in tetrad formalisms

The integration procedure [9], [19], [20] developed in the GHP formalism [17] — which is built on two intrinsic vectors or spinors picked out by the spacetime geometry — has recently been generalised [16], [13], [14], [15] to the GIF (generalised invariant formalism) [32], [33], [34]— which is built on one intrinsic vector or spinor picked out by the spacetime geometry. Compared to the familiar integration procedures in NP formalism [35], [9], this procedure is much more efficient and avoids detailed complicated gauge calculations which have the potential for errors. It also supplies the metric in a natural form, with coordinates chosen — as far as possible — in an intrinsic and invariant manner, which permits comparatively simple invariant classification procedure, equivalence problem analyses and symmetry investigations.

1.2 Equivalence problem and invariant classification of metrics

The equivalence problem is the problem of determining whether the metrics of two spacetimes are locally equivalent, and the original contribution of Cartan [5] directed attention to the Riemann tensor and its covariant derivatives up to $(q + 1)$th order, $\mathcal{R}^{q+1}$, calculated in a particular frame.

In going from $\mathcal{R}^q$ to $\mathcal{R}^{q+1}$ for a particular spacetime, if there is no new functionally independent Cartan scalar invariant and $\mathcal{R}^q$ and $\mathcal{R}^{q+1}$ have equal isotropy group, then all the local information that can be obtained about the spacetime is contained in the set $\mathcal{R}^{q+1}$. The set $\mathcal{R}^{q+1}$ is called the *Cartan scalar invariants* and provide the information for an *invariant classification* of the spacetime.

That the $q + 1$-derivatives are needed can be understood in the following way. Call the functionally independent elements I^α. We then have

$$dI^\alpha = I^\alpha{}_{|K} \omega^K.$$

Here $K = 1, 2, ..., \frac{n(n+1)}{2}$ numbers a basis on $F(M)$ and $I^\alpha_{|K} \equiv X_K(I^\alpha)$ where X_K is the dual basis to ω^K. Note that if a certain I^α appears in the qth derivative, then $I^\alpha_{|K}$ appears in the $q + 1$ derivative. In the case without symmetries the relation above can be inverted to give all ω^K and in particular the 1-forms spanning the tangent plane of M, ω^i, $i = 0, 1, ...n - 1$, in terms of the elements in $\mathcal{R}^{q+1}$. Hence the metric is obtained as $ds^2 = \eta_{ij}\omega^i\omega^j$. The proof can be extended to the case with symmetries [5], [21].

Two metrics are equivalent and represent the same spacetime if all their respective Cartan scalar invariants in $\mathcal{R}^{q+1}$ can be equated consistently. It is important to note that although there will be no new information about essential coordinates in the step from $\mathcal{R}^q$ to $\mathcal{R}^{q+1}$, there *may* be other new information, in particular about inconsistencies and also the nature of apparently non-redundant functions (including constants).

A complete invariant classification of a spacetime should include the true nature of apparently non-redundant functions, i.e., whether they are genuinely non-redundant, or can be transformed away by a coordinate transformation. A convenient way to do this for a particular spacetime is to carry out a simplified form of an equivalence problem: a metric with coordinates $x^i, i = 1, \ldots, 4$, and non-redundant functions $f^\alpha, \alpha = 1, \ldots m$, can be tested for equivalence against a second metric, which is its copy with exactly the same structure, but where we have relabelled the coordinates X^i and the non-redundant functions F^α. If, and only if, the equivalence problem confirms these two metrics as completely equivalent, and in particular identifies each of the apparently non-redundant functions with a Cartan invariant, then we can conclude that the non-redundant functions are essential and genuinely non-redundant. (Non-redundant constants can be treated as a special case of non-redundant functions, and are understood to be included in our discussions; but see [37] for a different treatment of non-redundant constants.) Note that when working with an explicit metric the $I^\alpha_{|K}$, may appear to "vanish" since they are explicitly calculated. However, if we want to compare two metrics each with an apparently non-redundant function, say $f(u)$ and $F(U)$ respectively, we do not only need the relation between them, but also between their derivatives: suppose that we find $F(U) = f(u)$ as the last coordinate relation at order p, then for consistency we need

$$\frac{dF}{dU} = \frac{dF}{du}\frac{du}{dU} = \frac{df}{du}\frac{du}{dU}.$$

When comparing this with the relation between $F_U(= dF/dU)$ and $f_u(= df/du)$, that will appear at order $p + 1$, we can then solve for $\frac{du}{dU}$.

1.3 Karlhede algorithm for invariant classification of metrics

A practical method for invariant classification was developed by Karlhede [21], using fixed frames. In this algorithm the number of functionally independent quantities is kept as small as possible at each step by putting successively the curvature and its covariant derivatives into canonical form, and only permitting

those frame changes which preserve the canonical form. In practice, rather than work directly with the Riemann tensor and its covariant derivatives, it is convenient to use decompositions of their spinor equivalents; a minimal set of such spinors has been obtained in [30]. The frame components of $\mathcal{R}^{q+1}$ in the canonical frame fixed by the Karlhede algorithm will be called the *Cartan-Karlhede scalar invariants*[1]. Although the Karlhede algorithm is more efficient than the original procedures proposed by Cartan or Brans [4], it may need to go as far as to $\mathcal{R}^7$ [21], and as a consequence, for some spacetimes, long complicated calculations are required, which usually need computer support, e.g., using the programme CLASSI [1], or the Maple-based GRTensor programme [18]. Subsequently, it has been shown how the Karlhede algorithm can be exploited to determine the structure of the isometry group of the spacetime, as well as subclasses within the spacetime which have additional isometries [22]; more recently the scheme has been the basis for an algorithm which determines whether a spacetime admits a homothetic Killing vector [25].

1.4 Invariant classification of metrics using GHP and GIF

The standard approach when calculating an invariant classification of a spacetime uses the Karlhede algorithm, exploits a minimal set of spinors, and the usual presentation is in the NP [35] formalism. However, recently, alternative formalisms have been shown to be more efficient in the performance of invariant classifications for certain classes of spacetimes.

In [6], [7], [8], an invariant classification procedure for (vacuum and non-vacuum) Petrov Type D spacetimes was discussed in the context of the GHP formalism: this formalism is particularly efficient when a pair of intrinsic spinors has been identified by the geometry, and calculations can be carried out in a spin- and boost-weighted scalar formalism, rather than restricting the spin and boost freedom in an ad hoc manner. The (weighted) invariant *GHP Cartan scalars* used for the invariant classification are closely related to, but not identical with the Cartan-Karlhede scalar invariants used in the Karlhede algorithm: the GHP Cartan scalars are frame derivatives of frame components of the Riemann spinor, whereas the Cartan-Karlhede invariant scalars are constructed by projecting the minimal set [30] of covariant derivatives of the Riemannn spinor onto the frame. It is also emphasised, that for invariant classifications using the GIF and GHP formalism, there is no requirement that the frame be fixed as much as possible at each step, unlike this requirement which is imposed in the Karlhede algorithm. In [34], [31] an invariant classification procedure for (vacuum and non-vacuum) Petrov Type N spacetimes was discussed in the context of GIF; this formalism is particularly efficient when one intrinsic spinor has been identified by the geometry, and calculations for an invariant classification can be carried out in a spin- and boost-weighted and null rotation invariant spinor formalism, rather than restricting the spin, boost and null rotation freedom of the frame in an ad

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[1]In the literature these are usually simply called the *Cartan scalar invariants*, but we wish to distinguish these invariants found using the Karlhede algorithm from other frame invariants.

hoc manner. Within the GIF the information for an invariant classification is carried by the *Cartan spinor invariants*. As soon as a second invariant spinor has been supplied within the GIF calculations, transfer can be made to the weighted scalar GHP formalism, and work continued in the GHP formalism using invariant GHP Cartan scalars, which are scalar versions of the Cartan spinor invariants.

1.5   CFPR spacetimes (excluding plane waves)

The subclass of CFPR (conformally flat pure radiation) spacetimes, which are not plane waves, has provided a very good illustration of the benefits of the GIF integration procedure [16]; furthermore, regarding their classification, Skea has also pointed out that "this application of the equivalence problem provides a non trivial didactic application of the use of the invariant classifications in practice" [40].

Integrating in the NP formalism, Wils [42] obtained a metric (containing one apparently non-redundant function of one coordinate) which was claimed to represent the whole class of CFPR spacetimes which were not plane waves; subsequently Koutras [23] showed that this was the first metric from which a new essential base coordinate was obtained at *third* order of the Cartan scalar invariants, which means that its invariant classification formally requires investigation of *fourth* order Cartan scalar invariants. Koutras and McIntosh [24] have given a slightly more general metric in different coordinates; this form includes plane waves as well as the Wils metric. When Edgar and Ludwig [10], [11], proposed what appeared to be a more general metric (containing three apparently non-redundant functions of one coordinate) to represent all CFPR spacetimes, this provided an ideal opportunity for a non-trivial application of the equivalence problem. (The nontrivial nature of the invariant classification of this metric was emphasised when Skea's investigation of it [40] revealed a bug in the CLASSI computer programme which is used to handle the complicated calculations in the usual NP form; furthermore, when the Edgar-Ludwig metric was used to test the new Maple based invariant classification package in GRTensor, problems occurred and the results of the invariant classification published in [39] are clearly in error, as Barnes [2] has pointed out.)

Skea [40] carried out a detailed investigation of the equivalence problem in Koutras-McIntosh coordinates for the Edgar-Ludwig and Wils metrics: this analysis confirmed that the Edgar-Ludwig spacetime is a genuine generalisation of the Wils spacetime, and all information about essential coordinates was obtained by third order, which means that the procedure will formally terminate with fourth order Cartan scalar invariants; Skea also found the particular subclass of the Edgar-Ludwig spacetime which coincided with the Wils spacetime, and as well he investigated the particular subclass of the Edgar-Ludwig spacetime which permitted one Killing vector.

Subsequently, Edgar and Vickers [16] have rederived the Edgar-Ludwig spacetime using GIF; the metric obtained is only slightly different in appearance than the versions in [10] or [11], but deeper examination confirms a more natural and

simpler structure, because of the fact that the coordinates have been chosen in an invariant and intrinsic manner. A striking illustration of this is that the Killing vector analysis of this GIF version of the Edgar-Ludwig metric is trivial [12].

It is therefore expected that an invariant classification of the version of the metric, obtained by GIF in intrinsic coordinates, will be particularly simple, efficient and transparent.

1.6 Outline

The first purpose of this present paper is to demonstrate how the simpler, more natural structure of the version of the Edgar-Ludwig spacetime derived in intrinsic coordinates by a GIF analysis [16], enables us to get a simpler and more transparent picture of its invariant classification; and in particular to investigate in full detail the role of the three apparently non-redundant functions, confirming that they are genuinely non-redundant. The second purpose is to supply the full details of a highly non-trivial example of an invariant classification, by working through the much more complicated calculations required for a complete invariant classification in Koutras-McIntosh coordinates of the Edgar-Ludwig spacetime; this example highlights a number of subtleties within the invariant classification procedure, as well as emphasising again, by comparison, the much simpler calculations for the intrinsic coordinate version. The third purpose is to supply the full details of a highly non-trivial example of an equivalence problem, by demonstrating equivalence between the respective invariant classifications in the intrinsic coordinates and Koutras-McIntosh coordinates of the Edgar-Ludwig spacetime.

The invariant classification — carried out in the intrinsic coordinates and tetrad derived by the GIF analysis [16] — is given in detail in Section 2. This completes and corrects the incomplete discussion by Edgar and Vickers of an invariant classification of this GIF version, in the last section in [16].

In Section 3 we give a classical invariant Karlhede classification of the GIF version of the Edgar-Ludwig spacetime in its intrinsic coordinates using the CLASSI programme. This demonstrates explicitly how the Karlhede algorithm requires a slightly different tetrad from that used in Section 2, although of course the basic results are the same, and the presentation is also simple and transparent.

Skea's results, [40] as quoted above for the Edgar-Ludwig spacetime, are easily confirmed in Sections 2 and 3, in principle: in general, all the essential coordinates are obtained at third order of the Cartan scalar invariants, which means that the classification process will formally end at fourth order, where additional information about the non-redundant functions *may* become available. Further investigation reveals that all three apparently non-redundant functions in the Edgar-Ludwig spacetime are genuine non-redundant functions, none of which can be transformed away by a coordinate transformation. Furthermore, for the generic class, all four essential base coordinates are supplied at third order, and the fourth order Cartan scalar invariants only repeat information supplied at a

lower level; but for one special subclass it is found that there are no new essential base coordinates after the three supplied at second order, and so for this subclass the algorithm formally ends at third order; and for a second special subclass it is found that all four essential base coordinates are supplied at third order, but that additional information about one of the non-redundant functions becomes available at fourth order. Moreover, this second special subclass which requires a fourth order Cartan scalar invariant, does not intersect with the Wils spacetime; hence the inequivalence of the Wils and the Edgar-Ludwig spacetimes is established at third order of the Cartan scalar invariants.

However, the last conclusion disagrees with some of those in [40], where it was argued that information from the *fourth* order Cartan scalar invariant $D^4\Phi_{56'}$ was needed to find all relations for which the Edgar-Ludwig spacetime reduced to the Wils spacetime. Therefore, in Section 4, we rework the calculations in [40], using CLASSI, and demonstrate how all relations are found already at third order. Incompatabilities might appear at the next order, but in this case all fourth order relations turn out to be identically satisfied.

In Section 5, we carry out an equivalence problem for the two different versions of the metric whose respective Karlhede classifications are in Sections 3 and 4. We work through the problem in complete detail until we obtain as the final result the explicit coordinate transformations between the two versions.

In Section 6, we illustrate how simple it is to investigate Killing vectors and homothetic Killing vectors in the version of the metric with intrinsic coordinates obtained by a GIF integration, compared with the version in Koutras-McIntosh coordinates.

The paper concludes with a summary and discussion.

2 Invariant classification of Edgar-Ludwig spacetime in intrinsic coordinates from GIF

Edgar and Vickers [16] have rederived all CFPR spacetimes, which are not plane waves, using GIF, obtaining in coordinates t, n, a, b the metric

$$g^{ij} = \begin{pmatrix} 0 & -1/a & 0 & 0 \\ -1/a & (-2s(t) + 2m(t)a + a^2 + b^2)/a & -n/a & -e(t)/a \\ 0 & -n/a & -1 & 0 \\ 0 & -e(t)/a & 0 & -1 \end{pmatrix} \qquad (1)$$

or equivalently

$$\begin{aligned} \mathrm{d}s^2 &= \Big(a\big(2s(t) - 2am(t) - a^2 - b^2\big) - e(t)^2 - n^2\Big)\mathrm{d}t^2 - 2a\mathrm{d}t\mathrm{d}n \\ &\quad + 2n\mathrm{d}t\mathrm{d}a + 2e(t)\mathrm{d}t\mathrm{d}b - \mathrm{d}a^2 - \mathrm{d}b^2 \end{aligned} \qquad (2)$$

where $m(t)$, $e(t)$, $s(t)$ are non-redundant functions of the coordinate t; this form includes the possibility of any of $m(t)$ or $e(t)$ or $s(t)$ being constant. We

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have the same notation as in [16], except that we replace $M(t)$, $E(t)$, $S(t)$ with respectively $m(t)$, $e(t)$, $s(t)$. This form represents the most general metrics for CFPR spacetimes (with zero cosmological constant); alternative equivalent forms are given in [10], [11]. In [16] the coordinate a was defined as a *positive* quantity by $a = \sqrt{\tau\bar{\tau}}$: however, it is easy to see that such a restriction is simply an artifact of that method, and can be removed; on the other hand, the restriction $a \neq 0$ is still valid.

In the final section of [16] explicit expressions for some GIF spinor versions of the Cartan invariants were given, and there was some discussion about the invariant classification for this spacetime in the context of these Cartan spinor invariants. We shall now complete these tables and discussion. We begin by repeating the zeroth, first and second order invariants quoted in [16].

At zeroth order, there is only the one Cartan spinor invariant

$$\Phi = \frac{q^2}{a} \tag{3}$$

At first order, there are four Cartan spinor invariants

$$\text{Þ}\Phi = 0, \qquad \eth\Phi = \frac{pq^2}{a^2}, \qquad \eth'\Phi = \frac{\bar{p}q^2}{a^2}, \qquad \text{Þ}'\Phi = -\frac{q^2}{a^2}\left(\frac{qn}{a} + 3p\boldsymbol{\iota} + 3\bar{p}\bar{\boldsymbol{\iota}}\right) \tag{4}$$

Here we are following GIF notation and conventions in [32] and [33]; in particular we are using the index-free "new compacted formalism" outlined in [32] and explained more fully in [33]: $\boldsymbol{\iota}$ is a second spinor which is generated in the GIF analysis, p and q are weighted scalar invariants which represent the spin and boost freedom; q is real while complex p satisfies $p\bar{p} = 1/2$. This is the same notation as in [16] except that we have replaced $\mathbf{I}$, P, Q, with $\boldsymbol{\iota}$, p, q respectively.

It is easy to see that we may invert (3) and (4) for a, p and q in terms of Cartan spinor invariants, and also for $(p\boldsymbol{\iota} + \bar{p}\bar{\boldsymbol{\iota}})$; therefore $\boldsymbol{\iota}$ is not uniquely determined, (and so neither is n) and — at this level — there clearly would remain the gauge freedom of a one parameter subgroup of null rotations. Since new information about the essential coordinates has arisen, we must go to the next order.

At second order, a complete set of independent Cartan spinor invariants is[2]

$$\begin{aligned}
\text{Þ}\eth\Phi &= 0, \qquad \eth\eth\Phi = \frac{2p^2q^2}{a^3}, \qquad \eth'\eth\Phi = \frac{q^2}{a^3}, \qquad \text{Þ}\text{Þ}'\Phi = 0 \\
\eth\text{Þ}'\Phi &= -\frac{pq^2}{a^3}\left(\frac{3qn}{a} + 6p\boldsymbol{\iota} + 8\bar{p}\bar{\boldsymbol{\iota}}\right) \\
\text{Þ}'\text{Þ}'\Phi &= \frac{q^4}{a^4}\left(-s(t) - 2am(t) - \frac{5}{2}a^2 + \frac{1}{2}b^2 + \frac{3n^2}{a}\right) \\
&\quad + \frac{3q^2}{a^3}\left(4p^2\boldsymbol{\iota}^2 + 5p\bar{p}\boldsymbol{\iota}\bar{\boldsymbol{\iota}} + \bar{p}^2\bar{\boldsymbol{\iota}}^2\right) + 12\frac{q^3n}{a^4}(p\boldsymbol{\iota} + \bar{p}\bar{\boldsymbol{\iota}})
\end{aligned} \tag{5}$$

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[2]Note the typo corrections in the second and last equations in (5).

together with complex conjugates. The GIF commutator equations enable us to concentrate on this reduced list of independent invariants.

We can now invert these equations and obtain explicit expressions for the spinor ι, as well as for n and the scalar combination $\left(-s(t) - 2am(t) + \frac{1}{2}b^2\right)$, in terms of Cartan invariants. Thus, at second order, if we make this choice of ι as our second spinor, we will have fixed the frame completely (there is no isotropy freedom remaining), and we also have determined three essential base coordinates. Moreover, making this choice of ι as the second dyad spinor enables us to transfer to the simpler GHP formalism (see [16] for a fuller discussion on when this is possible) since we require only the scalar parts of the remaining non-trivial Cartan spinor invariants, i.e., the GHP Cartan invariants. So we replace (5) with

$$\partial\partial\Phi \;=\; \frac{2p^2q^2}{a^3}, \qquad \partial'\partial\Phi = \frac{q^2}{a^3}, \qquad \partial\text{\th}'\Phi = -\frac{pq^2}{a^3}\left(\frac{3qn}{a}\right)$$

$$\text{\th}'\text{\th}'\Phi \;=\; \frac{q^4}{a^4}\left(-s(t) + am(t) + \frac{1}{2}a^2 + \frac{1}{2}b^2 + \frac{3n^2}{a}\right). \tag{6}$$

(The easiest way to obtain this table is to operate with the GHP scalars on the GHP version of (4).[3]) Again — since we have obtained new information about essential coordinates — we need to go one step further.

<u>At third order</u>, a comparison of the second order expressions with the zeroth and first order ones shows that the only possibly independent new information will come from the following GHP Cartan invariants

$$\partial\partial\text{\th}'\Phi \;=\; -12\frac{p^2q^2}{a^4}\frac{qn}{a} \tag{7}$$

$$\partial\text{\th}'\text{\th}'\Phi \;=\; \frac{pq^4}{a^5}\left(-4s(t) + 3am(t) + a^2 + 2b^2 + 15\frac{n^2}{a}\right) - \text{i}\frac{pq^4}{a^4}b \tag{8}$$

$$\text{\th}'\text{\th}'\text{\th}'\Phi \;=\; \frac{q^5}{a^6}\left(-as'(t) + a^2m'(t) - 9anm(t) + 10ns(t) + abe(t)\right.$$
$$\left. -4a^2n - 5nb^2 - 15\frac{n^3}{a}\right) \tag{9}$$

and complex conjugates; prime $'$ denotes differentiation with respect to t. The GHP commutator equations reduce the number of independent invariants.

<u>At fourth order</u>, a comparison of the third order expressions with the zeroth, first and second order ones shows that the only possibly new independent information will come from the operator $\text{\th}'$ acting on the scalar X defined by

$$X \;=\; \text{\th}'\text{\th}'\text{\th}'\Phi - \frac{q^5}{a^6}\left(-9anm(t) + 10ns(t) - 4a^2n - 5nb^2 - 15\frac{n^3}{a}\right)$$

$$\;=\; \frac{q^5}{a^6}\left(-as'(t) + a^2m'(t) + abe(t)\right) \tag{10}$$

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[3]The numerical coefficients in the GHP scalar invariants at this order and higher orders have a few corrections compared to the published version [3]: however, these details do not effect any arguments.

where we have excluded from X all terms whose behaviour under the operators we already know from lower orders. This gives

$$\mathbb{P}'X \quad = \quad \frac{q^6}{a^6}\Big(-s''(t) + am''(t) - 5\frac{ns'(t)}{a} + 4nm'(t) + be'(t)$$
$$-5\frac{nb}{a}e(t) + e(t)^2\Big). \tag{11}$$

All other fourth order GHP Cartan invariants can be easily seen to duplicate information already identified at lower orders.

We have already noted that by second order the spin and boost parameters p, q, and a second spinor $\boldsymbol{\iota}$ are all uniquely determined intrinsically, and so there is no isotropy freedom; this means that we can concentrate on the essential base coordinates. We have also noted that we can also solve for three essential base coordinates $a, n, \left(b^2 - 2s(t) - 4am(t)\right)$ at second order; but solving for a fourth coordinate is a little more complicated, since we have to go to third order, and the details will depend on the nature of the functions $s(t), m(t), e(t)$.

So we get the following cases:

Case (a) $m(t) \neq$ constant and/or $s(t) \neq$ constant
In this case, (8) and its complex conjugate can be solved for b, which gives a fourth essential coordinate; hence the third essential coordinate simplifies to $\left(s(t) + 2am(t)\right)$ (even further, by combining with the rest of (8) to obtain $s(t)$ or $m(t)$).

Case (b) $m(t) = m_0$, $s(t) = s_0$, $e(t) \neq$ constant , where m_0, s_0, are constants
In this case, (8) and its complex conjugate simply duplicate the third essential coordinate b, while (9) can be solved for $e(t)$ which gives a fourth essential base coordinate.

Case (c) $m(t) = m_0$, $s(t) = s_0$, $e(t) = e_0$, where m_0, s_0, e_0 are all constants
In this case, the last equation in (5) supplies a third essential base coordinate b^2. It is obvious that no new information about essential coordinates is obtained at third order, and so in this case, the procedure clearly terminates at third order.

For the first two **(a)**, **(b)**, of the above cases, new information about the essential coordinates was given at third order, and so we know that, in principle, we need to continue to fourth order where there _may_ be further information available. However, since all possible information about essential coordinates has already been obtained (no isotropy freedom remaining, and four base coordinates identified) clearly there can be no new information possible about _essential coordinates_, although there _may_ be new information about the nature of the apparently _non-redundant functions_.

Summing up, we have found that:

• if at least one of the functions $m(t), s(t), e(t)$, is not constant then all four essential base coordinates are obtained from GHP Cartan invariants at third order, and the procedure will therefore formally terminate at fourth order.

• when all of the functions $m(t), s(t), e(t)$ are constants, then only three essential base coordinates can be obtained from Cartan GHP invariants; these are

obtained at second order, and the procedure will therefore formally terminate at third order.

2.1 The three apparently non-redundant functions

We next investigate the three apparently non-redundant functions $m(t), s(t), e(t)$, and in particular their possible redundancy.

As noted in the introduction, a convenient way to study the invariant classification of a particular spacetime is to carry out an equivalence problem of the original spacetime with a relabelled version of itself. However, because of the very simple structure of this metric and the close relationship of its coordinates with its GHP Cartan invariants it has been easy to draw conclusions on its classification from a direct examination of its GHP Cartan invariants. Continuing in this manner, the three apparently non-redundant functions $m(t), s(t), e(t)$ can each be directly identified with a linear combination of the three GHP Cartan invariants $Þ'Þ'\Phi$, $\Re(\partial Þ'Þ'\Phi)$, $Þ'Þ'Þ'\Phi$, and hence are genuinely non-redundant, and cannot be transformed away.

The final step in the classification is to note that in **Cases (a), (b)**, since the coordinate t does not occur by itself (but only implicitly in the three functions), it will also be necessary to have identification of the (non-zero) derivatives $m'(t), s'(t), e'(t)$ with GHP Cartan invariants. In **Case (a)** this information is given in the third order GHP Cartan invariant $Þ'Þ'Þ'\Phi$, whereas in **Case (b)** this information is given in the fourth order Cartan spinor invariant $Þ'X$, (equivalently $Þ'Þ'Þ'Þ'\Phi$).

This completes the essentials of the classification procedure.

In [40], the classification of the Edgar-Ludwig metric in Koutras-McIntosh coordinates was carried out in the form of an equivalence problem with a relabelled version of itself, and this approach will be reworked in Section 4. Therefore in order to compare with this analogous investigation, we will now also treat this question of the nature of the three functions more formally and in full detail in the notation of an equivalence problem. In the usual manner, we consider a second spacetime whose line element is a direct copy of the first; in the second spacetime, we label the coordinates by T, N, A, B, the weighted scalars by P, Q, the second spinor by **I** and the non-redundant functions by $M(T), S(T), E(T)$. Because of the simple structure of the classification in these coordinates, we can immediately conclude from discussions above, and the identifications with Cartan invariants, the following equivalences,

$$N = n, \ A = a, \ B = b; \qquad P = p, \ Q = q; \qquad \mathbf{I} = \boldsymbol{\iota}$$

Furthermore, from (8) and the last equation in (5),

$$S(T) = s(t), \qquad M(T) = m(t). \tag{12}$$

When we substitute this information into the remaining third order Cartan invariant $Þ'Þ'Þ'\Phi$, we obtain

$$\frac{\mathrm{d}S(T)}{\mathrm{d}T} - 2a\frac{\mathrm{d}M(T)}{\mathrm{d}T} + bE(T) = \frac{\mathrm{d}s(t)}{\mathrm{d}t} - 2a\frac{\mathrm{d}m(t)}{\mathrm{d}t} + be(t) \tag{13}$$

A full analysis of these identities will involve separate consideration of constant and non-constant functions, as in the above cases. So taking each case separately:

Case (a) It follows from (12) that $t = t(T)$, and so from (13), by separating out the different coordinates,

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = \frac{\mathrm{d}M(T)}{\mathrm{d}T}, \qquad \frac{\mathrm{d}s(t)}{\mathrm{d}t} = \frac{\mathrm{d}S(T)}{\mathrm{d}T}. \tag{14}$$

Therefore, from (12) and (14), it follows that

$$\frac{\mathrm{d}T}{\mathrm{d}t} = 1, \qquad \text{and hence,} \quad T = t + t_o, \tag{15}$$

where t_0 is an arbitrary constant; subsequently from (12) (and (13)) it follows that

$$M(t + t_o) = m(t), \ \ S(t + t_o) = s(t), \ \big(E(t + t_o) = e(t)\big). \tag{16}$$

(Of course, it is obvious from (2) that t has the freedom of an arbitrary additive constant.) Furthermore, by direct substitution of all the earlier results for the two metrics in this case, we find that the equality for the only non-trivial fourth order GHP Cartan invariant (11) is identically satisfied.

Case (b) From (12) the functions $S(T), M(T)$ must also be constants, genuinely non-redundant, and equal to their counterparts, i.e., $S_0 = s_0$, $M_0 = m_0$. In this case, from (13) it follows that $E(T) = e(t)$ ($\neq e_0$) and hence $t = t(T)$. Since we have no other information at third order we must go to fourth order, where, after substituting all the equalities which we have found between the two metrics into (11), we obtain

$$\frac{\mathrm{d}E(T)}{\mathrm{d}T} = \frac{\mathrm{d}e(t)}{\mathrm{d}t} \tag{17}$$

but since we also know that $E(T) = e(t)$, then we can again deduce (15), and hence

$$E(t + t_0) = e(t) \tag{18}$$

Case (c) In this case also, from (12) the functions $S(T), M(T)$ must also be constants, and equal to their counterparts, i.e., $S_0 = s_0$, $M_0 = m_0$; and substitutions in (13) gives equality for the remaining constant function $E_0 = e_0$, so that all three constants, which have been identified with GHP Cartan invariants, are genuinely non-redundant and cannot be transformed away. This completes the consideration of all third order invariants, and we have already noted that for this case we do not need to consider the fourth order Cartan invariants. So all that remains is to identify the final (in this case non-essential) coordinates. To do this we compare the line element (2) with its direct copy in N, A, B, T coordinates; when we make the substitutions which we have just

identified, $N = n, A = a, B = b$, it follows trivially that $dT = dt$, and hence $T = t + t_0$ where t_0 is an arbitrary constant.

In all three cases, we have confirmed that there is complete compatability at the appropriate order, so that there is no redundancy between the three non-redundant functions $m(t), s(t), e(t)$, and none of them can be transformed away by a coordinate transformation (this is also true in the special cases of these functions being non-redundant constants); hence this spacetime cannot be presented in a form with less non-redundant functions than these three.

We noted earlier — for case **(c)** — that the equivalence problem is formally solved at third order, while — for cases **(a), (b)** — since we obtained information about essential coordinates at third order, then formally we need to go to fourth order to complete the equivalence problem, where there _may_ be new information about the non-redundant functions.. However, in the above calculations, we see that — in case **(a)** — we have been able to extract all possible information about the three non-redundant functions without needing to go to fourth order GHP Cartan invariants, whereas — in case **(b)** — we required one additional _fourth_ order GHP Cartan invariant (11).

This section amplifies the discussion on the invariant classification in the last section of [16], and corrects the brief comments in the last paragraph in that section. The fact that the two sets of coordinates and the two sets of non-redundant functions have been shown to be trivially identical by this analysis is of course due to the intrinsic nature of the coordinates and tetrad supplied by the GIF procedure.

2.2 Deducing the Karlhede classification

As noted in the Introduction, the general invariant classification procedure can be refined by the Karlhede classification algorithm whereby the frame is fixed as much as possible at each successive order of the algorithm. The classification which has just been performed has not followed that algorithm; in particular it was clear that at first order of GHP Cartan invariants the second dyad spinor could have been identified up to the freedom of a _one_ parameter null rotation, but instead we rather chose to identify the second dyad spinor completely at second order of GHP Cartan invariants. Therefore in order to refine the above classification to a Karlhede-type classification we need to make certain modifications.

We first need to rewrite $\boldsymbol{\iota} \to \boldsymbol{\iota} + q\bar{p}n\mathbf{o}/3a$ in the last equation in (4); this ensures that for the scalar part of this Cartan spinor invariant, $P'\Phi = 0$, which is the condition in the Karlhede algorithm. Additionally we must rewrite $\boldsymbol{\iota} \to \boldsymbol{\iota} + q\bar{p}n\mathbf{o}/3a$ in all higher order Cartan spinor invariants such as (5); this will lead to changes in the coefficients of any terms containing the n coordinate in the Cartan scalar invariants (6), (7), (8), (9), (11).

Furthermore, we have retained in the invariants the arbitrary spin and boost parameters, p, q respectively: however, for the zeroth order in these spacetimes the Karhede algorithm requires that $\Phi_{22} = 1$, which can be achieved with the

boost choice $q = a^{1/2}$; and to complete the standard canonical tetrad by making a real choice for $\partial\Phi_{22}$ can be achieved with the spin choice $p = 1/\sqrt{2}$.

There is, as we noted in the Introduction, one further difference between the GHP Cartan invariants quoted here from GIF and the standard Cartan-Karlhede invariants deduced from the Karlhede algorithm: the Cartan-Karlhede invariant scalars are constructed by projecting the minimal set of covariant derivatives of the Riemannn spinor onto the frame. Although, taking into account these various differences, we could now deduce the Cartan-Karlhede invariants by hand from the corresponding GHP Cartan scalars just calculated, we shall not write out these Cartan-Karlhede invariants explicitly here; instead we shall determine the Cartan-Karlhede invariants for this version of the spacetime by the usual CLASSI procedure, in the next section.

3 Karlhede classification by CLASSI of Edgar-Ludwig spacetime in intrinsic coordinates

In this section, for the line element (2) which was obtained in [16] we will make a coordinate change $a \to -c$ which gives the metric in coordinates t, n, c, b

$$
\begin{aligned}
\mathrm{d}s^2 &= \Big(-c\big(2s(t) + 2cm(t) - c^2 - b^2\big) - e(t)^2 - n^2\Big)\mathrm{d}t^2 + 2c\mathrm{d}t\mathrm{d}n \\
&\quad -2n\mathrm{d}t\mathrm{d}c + 2e(t)\mathrm{d}t\mathrm{d}b - \mathrm{d}c^2 - \mathrm{d}b^2
\end{aligned}
\tag{19}
$$

This is simply a technical change to simply the operations of the CLASSI programme in situations involving terms with square roots.

The line element (2) was obtained in [16] from a null tetrad ((67) in [16]) with arbitrary spin and boost parameters P, Q; more generally we now consider this tetrad with the additional complex parameter Z representing a null rotation (and rewriting P, Q as p, q respectively, with $p\bar{p} = 1/2$),

$$
\begin{aligned}
n &= \frac{q}{2c}\big(c^3 - 2c^2 m(t) + cb^2 - 2cs(t) - e^2(t) - n^2 + Z\overline{Z}\big)\mathrm{d}t \\
&\quad + q\mathrm{d}n + \frac{q}{c}\big(\Re\left(Z\right) - n\big)\mathrm{d}c + \frac{q}{c}\big(e(t) - \Im\left(Z\right)\big)\mathrm{d}b \\
l &= \frac{c}{q}\mathrm{d}t, \qquad \bar{m} = p\left(\overline{Z}\mathrm{d}t + \mathrm{d}c + i\mathrm{d}b\right), \qquad m = \bar{p}\left(Z\mathrm{d}t + \mathrm{d}c - i\mathrm{d}b\right)
\end{aligned}
$$

The parameters $p, q, \Re(Z), \Im(Z)$ are used to get the Riemann tensor and its derivatives in standard form (characterised by the functional dependence being minimised at each order). We begin with the choices $q = \sqrt{c}$ to get $\Phi_{22'} = 1$, and $\Re(Z) = 2n/3$ to get $D\Phi_{33'} = 0$[4]. Then $p = 1/\sqrt{2}$ makes $\Im(D\Phi_{23'}) = 0$ and finally, by putting $\Im(Z) = e(t)$, we obtain $\Im(D^2\Phi_{34'}) = 0$ (in this way the appearance of the last functional independent component is postponed to the third derivative). The frame is now completely specified and no isotropies remain.

---

[4]We use the abreviated notation for Cartan-Karlhede invariants used in [8], [40].

The Weyl tensor and curvature scalar are zero, so the Riemann tensor is given by the $\Phi_{ab'}$-spinor. A classification according to the Cartan-Karlhede procedure gives at zeroth and first order

$$\Phi_{22'} = 1, \qquad D\Phi_{23'} = -\frac{1}{\sqrt{2}c} \qquad (20)$$

Hence the first functionally independent quantity is found in the first derivative and c can be used as an essential coordinate.

At second order,

$$
\begin{aligned}
D^2\Phi_{24'} &= D^2\Phi_{33'} = \frac{1}{c^2}, \qquad D^2\Phi_{34'} = -\frac{\sqrt{2}n}{3c^{5/2}} \\
D^2\Phi_{44'} &= \frac{5}{2} - \frac{2m(t)}{c} - \frac{b^2}{2c^2} + \frac{s(t)}{c^2} + \frac{5n^2}{9c^3} \qquad (21)
\end{aligned}
$$

Two more functionally independent components are found and all isotropy is lost; clearly n can be used as a second essential coordinate; we could choose $D^2\Phi_{44'}$ as a third coordinate already at this stage, but the choice of essential coordinates get simpler if we also use the information from $D^3\Phi$.

At third order,

$$
\begin{aligned}
D^3\Phi_{25'} &= D^3\Phi_{34'} = \frac{-3}{\sqrt{2}c^3}, \qquad D^3\Phi_{35'} = D^3\Phi_{44'} = \frac{2n}{c^{7/2}} \\
D^3\Phi_{45'} &= \frac{\sqrt{2}}{c^4}\left(-\frac{37c^3}{10} + 3m(t)c^2 + b^2c - 2s(t)c - \frac{3n^2}{2}\right) + \mathrm{i}\frac{b}{\sqrt{2}c^2} \\
D^3\Phi_{55'} &= \frac{1}{c^{7/2}}\left(\frac{5nc^2}{3} - 2c^2m'(t) - cbe(t) - 2cnm(t) + cs'(t)\right. \\
&\qquad \left. - \frac{5b^2n}{3} + \frac{10ns(t)}{3} + \frac{5n^3}{3c}\right) \qquad (22)
\end{aligned}
$$

From $\Im(D^3\Phi_{45'})$ we can choose b as one of the essential coordinates.
From $D^2\Phi_{44'}$ we can then choose $m(t)$ or $s(t)$ as the last coordinate (if at least one is non-constant). If $m(t) = m_0$ and $s(t) = s_0$ both are constants, then $e(t)$ (providing it is not constant) in $D^3\Phi_{55'}$ can be used as the fourth coordinate. On the otherhand, if $e(t) = e_0$ also is a constant, then all coordinates are found in $D^2\Phi$. Note that in this case the third constant e_0 is found in $D^3\Phi$.
So therefore we can sum up in the same way as in the previous section,

• if at least one of the functions $m(t), s(t), e(t)$, is not constant then all four essential base coordinates are obtained from Cartan-Karlhede invariants at third order, and the procedure will therefore formally terminate at fourth order.

• when all of the functions $m(t), s(t), e(t)$ are constants, then only three essential base coordinates can be obtained from Cartan-Karlhede invariants; these are obtained at second order, and the procedure will therefore formally terminate at third order.

At fourth order,

$$D^4\Phi_{26'} = D^4\Phi_{35'} = D^4\Phi_{44'} = \frac{6}{c^4}, \qquad D^4\Phi_{36'} = D^4\Phi_{45'} = -\frac{6\sqrt{2}n}{c^{9/2}}$$

$$D^4\Phi_{46'} = \frac{1}{c^4}\left(\frac{74c^2}{5} - 12m(t)c - 5b^2 + 10s(t) + \frac{10n^2}{c}\right) - 4i\frac{b}{c^3}$$

$$D^4\Phi_{55'} = \frac{1}{c^4}\left(\frac{265c^2}{18} - 12m(t)c - 5b^2 + 10s(t) + \frac{10n^2}{c}\right)$$

$$D^4\Phi_{56'} = -\frac{161\sqrt{2}n}{18c^{5/2}} + \frac{4\sqrt{2}m'(t)}{c^{5/2}} + \frac{5be(t)}{\sqrt{2}c^{7/2}} + \frac{8\sqrt{2}nm(t)}{c^{7/2}} - \frac{5s'(t)}{\sqrt{2}c^{7/2}}$$

$$+\frac{35\sqrt{2}n(b^2 - 2s(t))}{6c^{9/2}} - \frac{20\sqrt{2}n^3}{3c^{11/2}} + i\frac{\sqrt{2}}{c^{7/2}}\left(\frac{e(t)c}{2} + \frac{5bn}{3}\right)$$

$$D^4\Phi_{66'} = 30 - 47\frac{m(t)}{c} + \frac{1}{c^2}\left(-\frac{19}{2}b^2 + 18m^2(t) + 29s(t) - 2m''(t)\right)$$

$$+\frac{1}{c^3}\left(11b^2m(t) - be'(t) - e^2(t) - 22m(t)s(t) + \frac{335}{18}n^2\right)$$

$$-4nm'(t) + s''(t)\Big) + \frac{1}{c^4}\left(\frac{5}{2}b^4 - 10b^2s(t) - 5bne(t) - 14n^2m(t)\right.$$

$$+5ns'(t) + 10s^2(t)\Big) + \frac{65n^2}{6c^5}\left(2s(t) - b^2\right) + \frac{145}{18}\frac{n^4}{c^6}$$

$$\Box(D^2\Phi_{44'}) = \frac{14}{c^2} \tag{23}$$

For completeness we have given all of the non-zero fourth order Cartan-Karlhede invariants [5]. When checking for equivalence with, e.g., a metric copy as in Section 2, it is essential that all Cartan-Karlhede invariants up to fourth order are consistent. In particular, since derivatives $s'(t), m'(t)$ appear in $\mathcal{R}^3$ and $e'(t)$ in $\mathcal{R}^4$, we need to ensure compatability of dT/dt which is obtainable from each of the three functions.

As noted earlier, the GHP Cartan scalars used for the classification in Section 2 are closely related to the Cartan-Karlhede scalars determined above; by comparing the two sets, we can see how the basic structures are the same, only the numerical details of the coefficients differ. As pointed out before, the main difference is due to the (slightly) different tetrads used, but also to the difference in definitions of GHP Cartan invariant scalars and Cartan-Karlhede scalars.

An investigation of the role of the non-redundant functions in this section simply duplicates the results in the previous section, and the details give once again the same three cases as in the previous section.

---

[5] In general the symmetrised derivatives are not sufficient for a complete classification. The additional quantities needed are given in [30]. For the present case the only non-zero of these is $\Box(D^2\Phi_{44'}) \equiv \Phi_{111'1';11'11}{}^{x'}{}_a{}^a{}_{x'} = \frac{14}{c^2}$.

4 Karlhede classification by CLASSI of Edgar-Ludwig spacetime in Koutras-McIntosh coordinates

Skea [40] investigated the equivalence of the Edgar-Ludwig spacetime given in [10], in the Koutras-McIntosh coordinates u, w, x, y

$$ds^2 = \Big(2f(u)x\big(h(u) + g(u)y + x^2 + y^2\big) - w^2\Big)du^2$$
$$+2xdudw - 2wdudx - dx^2 - dy^2 \qquad (24)$$

or equivalently,

$$g^{ij} = \begin{pmatrix} 0 & 1/x & 0 & 0 \\ 1/x & -2f(u)\big(h(u) + g(u)y + x^2 + y^2\big)/x & -w/x & 0 \\ 0 & -w/x & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (25)$$

and the Wils spacetime [42] given in the same coordinates but with $g(u) = 0 = h(u)$, in order to determine whether the Edgar-Ludwig spacetime could actually be reduced to the Wils spacetime by a coordinate transformation. We shall consider the more general problem of the complete invariant classification of the Edgar-Ludwig spacetime, in particular checking on the redundancy of the three apparently non-redundant functions; we shall then specialise these results to the equivalence problem of the Edgar-Ludwig and Wils spacetimes.

The Cartan-Karlhede scalar invariants for the Edgar-Ludwig spacetime in Koutras-McIntosh coordinates are given to fourth order in [40], in terms of the canonical tetrad obtained there, and we shall also use this tetrad and quote directly the Cartan-Karlhede scalar invariants in [40].

In [40] it was argued that the fourth order invariant $D^4\Phi_{56'}$ is needed to show inequivalence. However, in the present work we find that this can rather be achieved already at third order.

As Skea points out, Cartan-Karlhede scalar invariants of first and second order supply three base coordinates directly, $D\Phi_{23'}$, $D^2\Phi_{34'}$, $D^2\Phi_{44'}$,

$$\Phi_{22'} = 1 \qquad (26)$$

$$D\Phi_{23'} = \frac{-1}{\sqrt{2}x} \qquad (27)$$

$$D^2\Phi_{34'} = \frac{xf_u - 2wf}{6f^{3/2}x^{5/2}} \qquad (28)$$

$$D^2\Phi_{44'} = -\frac{g(u)y + h(u)}{2x^2} + \frac{5}{2} - \frac{y^2}{2x^2}$$
$$+\frac{5\big(f(u)^2w^2 - f(u)wxf_u - 2x^2f_u{}^2\big)}{18f(u)^3x^3} + \frac{f_{uu}}{2f(u)^2x} \qquad (29)$$

But instead of $D^2\Phi_{44'}$ we shall prefer to begin with the simpler third order

Cartan-Karlhede scalar invariant,

$$\Im(D^3\Phi_{45'}) = \frac{2y + g(u)}{2\sqrt{2}x^2}$$

We shall now follow the usual convention, used also by Skea, that coordinates x, y, w, u and non-redundant functions $f(u), g(u), h(u)$ in the first spacetime have direct counterparts X, Y, W, U and $F(U), G(U), H(U)$ in a second space-time.

So if we begin with the three simplest Cartan-Karlhede scalar invariants, we find the following equivalences

$$D\Phi_{23'}: \qquad\qquad x = X \qquad\qquad\qquad (30)$$

$$\Im(D^3\Phi_{45'}): \qquad\qquad y + g(u)/2 = Y + G(U)/2 \qquad\qquad (31)$$

$$D^2\Phi_{34'}: \qquad\qquad \frac{xf_u - 2wf}{f^{3/2}} = \frac{XF_U - 2WF}{F^{3/2}} \qquad\qquad (32)$$

where (30) has been used to obtain (31) and (32); at this stage we have identified three essential base coordinates, and it is obvious that they will always be functionally independent.

Going next to the more complicated relation for $D^2\Phi_{44'}$ in [40], we rearrange

$$
\begin{aligned}
D^2\Phi_{44'} &= -\frac{g(u)y + h(u)}{2x^2} + \frac{5}{2} - \frac{y^2}{2x^2} + \frac{5\big(f(u)^2 w^2 - f(u)wxf_u - 2x^2 f_u{}^2\big)}{18f(u)^3 x^3} \\
&\quad + \frac{f_{uu}}{2f(u)^2 x} \\
&= \frac{5}{2} - \frac{\big(y + g(u)/2\big)^2 + h(u) - g(u)^2/4}{2x^2} \\
&\quad + \frac{5\Big(\big(f(u)w - xf_u/2\big)^2 - 9x^2 f_u^2/4\Big)}{18f(u)^3 x^3} + \frac{f_{uu}}{2f(u)^2 x} \qquad (33)
\end{aligned}
$$

Now, making use of (30), (31) and (32), we deduce from (33) that

$$F^3 f^3\big(4h(u) - g(u)^2\big) + 5x(f^3 F_U^2 - F^3 f_u^2) + 4xfF(F^2 f_{uu} - f^2 F_{UU}) = 0. \quad (34)$$

We next consider $\Re(D^3\Phi_{45'})$ and by a similar calculation, we find that

$$
\begin{aligned}
-4F^3 f^3\big(4h(u) - g(u)^2\big) + 15x(f^3 F_U^2 - F^3 f_u^2) \\
+ 12xfF(F^2 f_{uu} - f^2 F_{UU}) = 0. \qquad (35)
\end{aligned}
$$

Together (34), (35) give the key equations

$$(ff_{uu} - 5f_u^2/4)f^{-3} = (FF_{UU} - 5F_U^2/4)F^{-3}, \qquad (36)$$

$$h(u) - g(u)^2/4 = H(U) - G(U)^2/4 \qquad (37)$$

18

From (36) we can identify a fourth essential base coordinate — in general, but only providing $(ff_{uu} - 5f_u^2/4)f^{-3} \neq$ constant; this means that we also will need to consider separately the case $(ff_{uu} - 5f_u^2/4)f^{-3} = f_0 = (FF_{UU} - 5F_U^2/4)F^{-3}$, where f_0 is constant. Further analysis will involve (37), and again we will need to consider separate cases $h(u) - g(u)^2/4 = h_0 = H(U) - G(U)^2/4$ where $h_0 =$ constant, and $\neq$ constant.

The only remaining third order Cartan-Karlhede scalar invariant where we can get any independent information is $D^3\Phi_{55'}$.[6] We now carry out a similar rearrangement as we did above for $D^2\Phi_{44'}$,

$$
\begin{aligned}
2\sqrt{2}D^3\Phi_{55'} \;=\; & -\frac{g_u y + h_u}{f^{1/2}x^{5/2}} \\
& +\frac{5}{3x^{7/2}}\Big((2fw - xf_u)/f^{3/2}\Big)\Big(x^2 - (y + g/2)^2 - h + g^2/4\Big) \\
& +\frac{5}{24x^{9/2}}\Big((2fw - xf_u)/f^{3/2}\Big)^3 \\
& +\frac{1}{8x^{5/2}f^3}\Big((2fw - xf_u)/f^{3/2}\Big)\Big(4ff_{uu} - 5f_u^2\Big) \\
& +\frac{1}{24x^{3/2}f^{9/2}}\Big(24f^2 f_{uuu} + 90f_u^3 - 108ff_u f_{uu}\Big) \qquad (38)
\end{aligned}
$$

and, making use of (30) and (32), as well as (37), (36), we deduce

$$
4x^{-1}(g_u y + h_u)f^{-1/2} - \Big(15f_u^3 + 4f^2 f_{uuu} - 18ff_u f_{uu}\Big)f^{-9/2}
$$
$$
= 4x^{-1}(G_U Y + H_U)F^{-1/2} - \Big(15F_U^3 + 4F^2 F_{UUU} - 18FF_U F_{UU}\Big)F^{-9/2}. \quad (39)
$$

Now using (31) gives

$$
4x^{-1}y(g_u f^{-1/2} - G_U F^{-1/2})
$$
$$
+4x^{-1}\Big(h_u f^{-1/2} - H_U F^{-1/2} - (gg_u f^{-1/2} - GG_U F^{-1/2})/2\Big)
$$
$$
-\Big(15f_u^3 + 4f^2 f_{uuu} - 18ff_u f_{uu}\Big)f^{-9/2}
$$
$$
= -\Big(15F_U^3 + 4F^2 F_{UUU} - 18FF_U F_{UU}\Big)F^{-9/2} \qquad (40)
$$

A quick examination of the fourth order Cartan-Karlhede scalar invariants listed in [40], shows that most give duplications of lower order information[7]. For the remainder: $\Im(D^4\Phi_{56'})$ also requires a little manipulation to show that no new information is available[8]; $\Re(D^4\Phi_{56'})$ requires a little more manipulation to show

---

[6]Note that in the explicit expression for $D^3\Phi_{55'}$ on page 2396 of [40] there is a typographical error; the first term on the right hand side should contain the factor $(xf_u - 2fw)$, and not $(x - 2fw)$. $D^3\Phi_{55'}$ is given correctly in [39].

[7]Note that in the explicit expression for $D^4\Phi_{55'}$ on page 2397 of [40] the coefficient 265 should be 256.

[8]The Cartan-Karlhede scalar invariant $\Im(D^4\Phi_{56'})$ is missing the additional term $\frac{5wg}{6x^{7/2}f^{1/2}}$

that no new information is available[9]; the only fourth order Cartan-Karlhede scalar invariant which has new information is $D^4\Phi_{66'}$.[10] We now carry out a similar rearrangement as we did above for $D^2\Phi_{44'}$ and $D^3\Phi_{55'}$, and making use of all earlier results, obtain

$$f_{uuuu}/(4f^3) - 135f_u^4/(128f^6) - 45f_u^3 w/(16xf^5)$$
$$-9f_{uu}f_u^2/(4f^5) + 27f_{uu}f_u w/(8f^4 x) - 11f_{uuu}f_u/(4f^4)$$
$$-3f_{uuu}w/(4xf^3) + (g_u y + h_u)f_u/(8xf) - (g_{uu}y + h_{uu})/(4fx)$$
$$= F_{UUUU}/(4F^3) - 135F_U^4/(128F^6) - 45F_U^3 W/(16XF^5)$$
$$-9F_{UU}F_U^2/(4F^5) + 27F_{UU}F_U W/(8F^4 X) - 11F_{UUU}F_U/(4F^4)$$
$$-3F_{UUU}W/(4XF^3) + (G_U Y + H_U)F_U/(8XF) - (G_{UU}Y + H_{UU})/(4XF) \quad (41)$$

It is clear that a complete invariant classification of this spacetime in these coordinates will involve a generic case, and a number of special cases.

So we now look at each of these various cases individually, and in full detail.

Case (A) $(ff_{uu} - 5f_u^2/4)f^{-3} \neq$ constant.

For this case, (36) gives a fourth essential base coordinate U, which is a function of u alone. We have found four essential base coordinates in $\mathcal{R}^3$; so the algorithm will formally terminate at $\mathcal{R}^4$.

Case (B) $(ff_{uu} - 5f_u^2/4)f^{-3} = f_0$, constant; $h(u) - g(u)^2/4 \neq$ constant.

For this case, from (37) we get a fourth essential base coordinate U, which is a function of u alone. Again, we have found four essential base coordinates in $\mathcal{R}^3$; so the algorithm will formally terminate at $\mathcal{R}^4$.

Case (C) $(ff_{uu} - 5f_u^2/4)f^{-3} = f_0$, constant; $h(u) - g(u)^2/4 = h_0$ constant.

There are only three essential base coordinates at this stage; but when these constants are substituted into (40) we obtain

$$F^{-1/2}G_U = f^{-1/2}g_u \quad (42)$$

which supplies a fourth essential base coordinate — but only providing that $f^{-1/2}g_u \neq$ constant.

Now we have to split into two subcases:

(i) If $f^{-1/2}g_u \neq$ constant, then from (42) we have a fourth essential coordinate, and can conclude $U = U(u)$. Again, we have found four essential base coordinates in $\mathcal{R}^3$; so the algorithm will formally terminate at $\mathcal{R}^4$.

(ii) If $f^{-1/2}g_u = g_0$, constant, then we have no fourth essential coordinate. We note that no expressions containing the u coordinate occur in the Cartan-Karlhede scalar invariants; hence there is no fourth base coordinate for this special case, in $\mathcal{R}^3$. In fact, for this special case, it can be seen that the three essential base coordinates are all given in $\mathcal{R}^2$ in (27), (28), (29), and since there

---

[9]Note that in the explicit expression for $D^4\Phi_{56'}$ on page 2397 of [40] there is a typographical error of the same type as in $D^3\Phi_{55'}$; the first term on the right hand side should contain the factor $(2fw - xf_u)$, and not $(2fw - x)$. The term in w^3 should have a denominator with $x^{11/2}$ instead of $x^{1/2}$. The numerical coefficient 63 should instead be 36.

[10]The expression with only the term x^4 in denominator should instead have $4x^4$.

is no further information about essential coordinates supplied at third order, the procedure terminates at $\mathcal{R}^3$ for this special case.

4.1 The three apparently non-redundant functions.

Investigations of the nature of the three apparently non-redundant functions in this coordinate system are more complicated and involved than in the intrinsic coordinates earlier. We will consider each case separately in detail:

Case (A)

The identification of a fourth essential base coordinate $U = U(u)$ enables us to separate out those terms with different coordinates in (40), to get

$$
\left(15f_u^3 + 4f^2 f_{uuu} - 18ff_u f_{uu}\right)f^{-9/2}
$$
$$
= \left(15F_U^3 + 4F^2 F_{UUU} - 18FF_U F_{UU}\right)F^{-9/2} \tag{43}
$$

$$
f^{-1/2}g_u = F^{-1/2}G_U, \tag{44}
$$

$$
f^{-1/2}h_u = F^{-1/2}H_U + (g - G)G_U F^{-1/2}/2. \tag{45}
$$

Moreover, when we differentiate (36) with respect to the coordinate $U(u)$ we obtain

$$
\left(15f_u^3 + 4f^2 f_{uuu} - 18ff_u f_{uu}\right)f^{-4}\frac{\mathrm{d}u}{\mathrm{d}U}
$$
$$
= \left(15F_U^3 + 4F^2 F_{UUU} - 18FF_U F_{UU}\right)F^{-4}. \tag{46}
$$

Comparing (46) with (43) gives

$$
\frac{\mathrm{d}u}{\mathrm{d}U} = \frac{F^{1/2}}{f^{1/2}}. \tag{47}
$$

The equation (36) can now be rearranged, using (47), into a second order differential equation for $F(U)$ in terms of $f(u(U))$, with the solution

$$
F(U) = f(u(U))/(c_0 + c_1 U)^4, \qquad c_1, c_0 \text{ are constants} \tag{48}
$$

From (48) it follows that

$$
\frac{\mathrm{d}u}{\mathrm{d}U} = (c_0 + c_1 U)^{-2}, \tag{49}
$$

so that, from (49),

$$
u(U) = c_2 - \frac{1}{c_1(c_0 + c_1 U)}, \text{ for } c_1 \neq 0, \quad \text{and} \quad u(U) = c_2 + \frac{U}{c_0^2}, \text{ for } c_1 = 0 \tag{50}
$$

for c_2 constant; and hence

$$
\begin{aligned}
F(U) &= \frac{1}{(c_0 + c_1 U)^4} f\left(c_2 - \frac{1}{c_1(c_0 + c_1 U)}\right), \text{ for } c_1 \neq 0, \\
F(U) &= \frac{1}{c_0^4} f\left(c_2 + \frac{U}{c_0^2}\right), \text{ for } c_1 = 0.
\end{aligned}
\tag{51}
$$

In addition, using (47) in (44), we get

$$
G_U = g_U, \quad \text{and hence,} \quad G(U) = g(u(U)) + c_3, \text{ where } c_3 \text{ is constant,} \tag{52}
$$

and where we can replace $u(U)$ with the two possibilities in (50).
There remains still some more information in (45): combining with (47)

$$
H_U = h_U - (g - G)G_U/2, \tag{53}
$$

but it is easy to see that (53) is just the U derivative of (37); therefore, from (37),

$$
H(U) = h(u(U)) + c_3^2/4 + c_3 g(u(U))/2 \tag{54}
$$

where we can replace $u(U)$ with the two possibilities in (50).
Clearly since $f(u), g(u), h(u)$ and $F(U), G(U), H(U)$ are completely arbitrary functions, then (51), (52), (54) respectively can always be satisfied for a given choice of $f(u), g(u), h(u)$ or $F(U), G(U), H(U)$. According to the theory, incompatability might appear at next order; but a direct check confirms that the fourth order conditions are identically satisfied, and hence we have equivalence.

Case (B)
In this case $F(U)$ can be obtained directly, since from (36) we obtain also $(FF_{UU} - 5F_U^2/4)F^{-3} = f_0$. These can be integrated to give [40]

$$
f(u) = \frac{c_0^2}{(c_0^2(u - u_0)^2 - f_0)^2} \text{ for } c_0, u_0 \text{ constants,} \tag{55}
$$

$$
F(U) = \frac{C_0^2}{(C_0^2(U - U_0)^2 - F_0)^2} \text{ for } C_0, U_0 \text{ constants.} \tag{56}
$$

As in the previous case, since $U = U(u)$ we can separate (40) to get (44) and (45).
(Condition (43) is trivially satisfied, since, in this case, we have the trivial identity,

$$
15f_u^3 + 4f^2 f_{uuu} - 18ff_u f_{uu} = 0 = 15F_U^3 + 4F^2 F_{UUU} - 18FF_U F_{UU}. \text{)}
$$

The derivative with respect to the fourth coordinate $U = U(u)$ of (37) is

$$
(4H_U - 2GG_U) = (4h_u - 2gg_u)\frac{\mathrm{d}u}{\mathrm{d}U} \tag{57}
$$

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and comparing with (45) gives again (47); and so, from (55), (56) we can find the relationship between the fourth pair of coordinates from

$$c_0 \int \frac{\mathrm{d}u}{c_0^2(u-u_0)^2 - f_0} = C_0 \int \frac{\mathrm{d}U}{C_0^2(U-U_0)^2 - f_0} \ , \tag{58}$$

where there will be different explicit integrals depending on the nature of f_0. Also from (47), together with (44), we obtain again (52), and therefore, as in the generic Case **(A)**, we obtain $G(U), H(U)$ from (52) and (54) respectively, where we can replace $U(u)$ from the results in (58).

Clearly since $f(u), g(u)$ and $F(U), G(U)$ are completely arbitrary functions, then (51), (52), respectively can always be satisfied for a given choice of $f(u), g(u)$ or $F(U), G(U)$; furthermore, for the special case of $f(u)$ defined by (55), the corresponding $F(U)$ is defined by (56). Once again the fourth order conditions are identically satisfied and we have equivalence.

Case (C)

In this case — as in Case **(B)** — $F(U)$ and $f(u)$ can be obtained directly, and are given by (55) and (56). We also obtain directly

$$h(u) - g(u)^2/4 = h_0 = H(U) - G(U)^2/4 \tag{59}$$

for h_0 an arbitrary constant.

(i) $f^{-1/2}g_u \neq$ constant

For this subcase, we cannot get any more direct information in $\mathcal{R}^3$; so we substitute all the above identities for this subcase in the fourth order invariant (41), and after a long simplification obtain

$$F^{-1}G_{UU} - F^{-2}F_U G_U/2 = f^{-1}g_{uu} - f^{-2}f_u g_u/2. \tag{60}$$

Moreover, when we differentiate (42) with respect to the fourth coordinate $U = U(u)$, we get

$$F^{-1/2}G_{UU} - F^{-3/2}F_U G_U/2 = \left(f^{-1/2}g_{uu} - f^{-3/2}f_u g_u/2\right)\frac{\mathrm{d}u}{\mathrm{d}U}, \tag{61}$$

and so we once again obtain (47), and hence from (55) and (56) we can obtain again the relationship (58) between the fourth coordinates. In addition, (47), together with (44), leads again to (52), where we can replace $U(u)$ from the results in (58).

Clearly since $g(u)$ and $G(U)$ are completely arbitrary functions, then (52) can always be satisfied for a given choice of $g(u)$ or $G(U)$; furthermore, for the special case of $f(u), h(u)$ defined by (55), (59) the corresponding $F(U), H(U)$ are given by (56), (59). In this particular case, we do actually get new information from the fourth order Cartan-Karlhede invariants; one non-trivial *fourth* order condition (41) was needed to complete the invariant classification via (60) in order to complete the information about the one remaining non-redundant function, $G(U)$, but the remaining fourth order invariants are identically satisfied. Hence we have equivalence.

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(ii) $f^{-1/2}g_u = g_0$, constant

For this subcase, since the functions $f(u), F(U)$ are given by (55) and (56), we can use these to get immediately

$$g(u) = c_0 g_0 \int \frac{\mathrm{d}u}{c_0^2(u-u_0)^2 - f_0} \; , \; G(U) = C_0 g_0 \int \frac{\mathrm{d}U}{C_0^2(U-U_0)^2 - f_0} \qquad (62)$$

where there will be different explicit integrals depending on the nature of f_0; these are quoted explicitly in [40]. The functions $h(u)$ and $H(U)$ follow from (59).

As noted above, for this special subcase, the three essential base coordinates are given in $\mathcal{R}^2$, and the procedure formally terminates in $\mathcal{R}^3$. So all that remains is to relate the fourth pair of (non-essential) coordinates. Note that this cannot be found from $\mathcal{R}^3$, so we compare the line element (24) with its direct copy in U, W, X, Y coordinates; when we make the substitutions which we have just identified,

$$x = X, \qquad y = Y + c_3/2, \qquad w = \frac{f^{1/2}}{2}\left(\frac{2W}{F^{1/2}} + x\left(\frac{f_u}{f^{3/2}} - \frac{F_U}{F^{3/2}}\right)\right) \qquad (63)$$

where $c_3 = G(U) - g(u) = $ constant. It follows, after considerable simplification, that

$$\frac{\mathrm{d}U}{\mathrm{d}u} = \frac{f^{1/2}}{F^{1/2}} \qquad (64)$$

as in the other cases; since $F(U), f(u)$ are known in this special case, this yields again (58).

For this special subcase, the functions $f(u), g(u), h(u)$ defined by (55), (62), (59) have the corresponding functions $F(U), G(U), H(U)$ given by (56), (62), (59). The three essential coordinates have been identified in $\mathcal{R}^2$, and the functional relationships are given in $\mathcal{R}^3$ where there is complete compatability. Hence there is equivalence also for this special subclass.

So overall there is equivalence in all three cases, with the apparently non-redundant functions (including the special cases when they are non-redundant constants) genuinely non-redundant; furthermore, there was only one subcase, Case **(C(i))** which required explicit use of fourth order Cartan scalar invariants.

The coordinate freedom for the metric in these coordinates is, in general, given by (63) with the fourth coordinate pair having different transformations depending on the different cases: (50) for Case **(A)**, and (58) for Cases **(B)**, **(C)**, and hence also different relationships, for these different cases, between the respective sets of non-redundant functions.

There was not the direct trivial identifications between the coordinates and the sets of non-redundant functions such as we obtained in the intrinsic coordinate version in the previous sections; this is because the Koutras-McIntosh coordinates and tetrad have less of an intrinsic character.

4.2 Equivalence problem for the Edgar-Ludwig and Wils spacetimes

In order to specialise the work in this section to the equivalence problem for the Edgar-Ludwig and Wils spacetimes, we put $H(U) = 0 = G(U)$ for the latter. It is obvious from the above calculations, that inequivalence can be deduced at an early stage in the above calculations — in fact from equation (37); this is in $\mathcal{R}^3$. (Although a fourth order invariant gave new information in the invariant classification of the Edgar-Ludwig spacetime — for the special subclass Case **(C(i))** — this subclass does not intersect with the Wils metric.)

Of course the Edgar-Ludwig spacetime with $h(u) = 0 = g(u)$ coincides with the Wils metric, but, in view of the fact that there was a non-trivial identification between the respective sets of non-redundant functions, we would expect a more general subclass of the Edgar-Ludwig spacetime with the property of the Wils spacetime.

As well as equation (37), which we have just noted, which imposes

$$h(u) = g(u)^2/4 \tag{65}$$

we have to consider equation (36),

$$(ff_{uu} - 5f_u^2/4)f^{-3} = (FF_{UU} - 5F_U^2/4)F^{-3}. \tag{66}$$

These two conditions imply that the subclass must be in Case **(A)** and Case **(C(ii))**.

From (44) (Case **(A)**) and (42) (Case **(C(ii))**) it immediately follows that

$$g_u = 0, \quad \text{hence} \quad g(u) \text{ and } h(u) \text{ are constants.} \tag{67}$$

When these results are substituted, the remaining second and third order Cartan-Karlhede invariants are identically satisfied. Further substitutions show that all fourth order Cartan-Karlhede invariants are identically satisfied. This confirms Skea's result [40] that the Edgar-Ludwig spacetime only reduces to the Wils metric when $g(u)$ and $h(u)$ are constant functions; but it has also been shown here that this conclusion can be obtained by the *third order* of Cartan-Karlhede invariants.

4.3 Comparison with Skea's approach

Finally we will compare the details of the above analysis with the arguments in [40]. Early in the analysis, it was argued that itwas necessary to use the fourth order Cartan-Karlhede scalar invariant $\Im(D^4\Phi_{56'})$ for an invariant analysis of the Edgar-Ludwig spacetime, and to solve the equivalence problem between this spacetime and the Wils spacetime; however, we have shown that this invariant is not independent of lower order invariants, and gives no new information.

In his analysis of equation (36), Skea concentrates on the generic case and does not consider explicitly the possibility of $(ff_{uu} - 5f_u^2/4)f^{-3} = f_o$, constant,

and other subcases defined by constant functions — at that stage. However in order to consider the Killing vector case (our Case **(C(ii))**), he is led to the equivalent equation $15f_u^3 + 4f^2 f_{uuu} - 18ff_u f_{uu} = 0$, and to other equations defining subcases. Subsequently he presents a table identifying the three cases corresponding to our Cases **(A), (B), (C(i))**.

5 Equivalence problem between metric in intrinsic coordinates and Koutras-McIntosh coordinates

We shall now solve the equivalence problem for the Edgar-Ludwig spacetime given in the two different coordinate systems, using the respective invariant classifications in Sections 3 and 4. The complete result will give the explicit coordinate transformations between the two systems.

The metrics (19) and (24) should be equivalent and we now proceed to show this by comparing their invariant classifications, i.e., by solving the set of equations

$$D^n \Phi_{ij'} = D^n \tilde{\Phi}_{ij'} \tag{68}$$

where $\Phi_{ij'}$ and $\tilde{\Phi}_{ij'}$ refer to the two different metrics respectively.

<u>At first order,</u> from $D\Phi_{23'}$ and $D\tilde{\Phi}_{23'}$ it follows that

$$c = x. \tag{69}$$

<u>At second order,</u> from $D^2 \Phi_{34'}$ and $D^2 \tilde{\Phi}_{34'}$ it follows that

$$n = \frac{2fw - xf_u}{(2f)^{3/2}}. \tag{70}$$

One also obtains from $D^2 \Phi_{44'}$ and $D^2 \tilde{\Phi}_{44'}$,

$$\frac{4ff_{uu} - 5f_u^2}{4f^3}x + b^2 - \left(y + \frac{g}{2}\right)^2 + \frac{g^2}{4} - h + 4mx - 2s = 0 \tag{71}$$

<u>At third order,</u> from $\Im(D^3 \Phi_{45'})$ and $\Im D^3(\tilde{\Phi}_{45'})$ it follows that

$$b = y + \frac{g}{2}. \tag{72}$$

At this stage we have related three pairs of essential coordinates.
From $\Re(D^3 \Phi_{45'})$ together with $D^2 \Phi_{44'}$ and their counterparts, one then gets

$$m(t) = \frac{5f_u^2 - 4ff_{uu}}{16f^3}, \qquad s(t) = -\frac{(h - g^2/4)}{2} \tag{73}$$

Making use of the above results, from $D^3\Phi_{55'}$ and $D^3\Phi_{55'}$ and their counterparts, we obtain

$$8\sqrt{2}\Big(-2x^2 m'(t) - x\big(y + g/2\big)e(t) + xs'(t)\Big)$$
$$= -4(g_u y + h_u)xf^{-1/2} + x^2\Big(15f_u^3 + 4f^2 f_{uuu} - 18ff_u f_{uu}\Big)f^{-9/2} \qquad (74)$$

A full analysis of these identities will involve separate consideration of constant and non-constant functions, as in the previous sections. So looking at the three cases from Section 4, in turn:

Case (A) $(ff_{uu} - 5f_u^2/4)f^{-3} \neq f_0$, constant.
For this case, from (73) we deduce a fourth essential coordinate t, which is a function of u alone, and from (74), by separation it follows that

$$m'(t) = -\Big(15f_u^3 + 4f^2 f_{uuu} - 18ff_u f_{uu}\Big)f^{-9/2}/16 \qquad (75)$$

$$s'(t) = f^{-1/2}(2gg_u - 4h_u), \qquad (76)$$

$$e(t) = g_u/2\sqrt{2}f^{1/2}. \qquad (77)$$

Furthermore, by differentiating (73) with respect to t, and comparing with (75), we obtain

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{\sqrt{2}f^{1/2}}, \qquad \text{and hence,} \quad t = \sqrt{2}\int f(u)^{1/2}\mathrm{d}u, \qquad (78)$$

from which we can invert and get the explicit relationship $u = u(t)$ for any function $f(u)$. The functions $m(t), s(t), e(t)$ are then determined by making the substitution $u = u(t)$ into the equations (73), (77). As noted above, formally we need to go one order further, but it is easy to confirm that all the Cartan-Karlhede invariants are identically satisfied at fourth order.

Case (B) $(5f_u^2 - 4ff_{uu})f^{-3} = f_0$, constant; $h(u) - g(u)^2/4 \neq$ constant.
This, as we saw in Section 4, corresponds to

$$f(u) = \frac{c_o^2}{(c_0^2(u - u_0)^2 - f_0)^2} \text{ for } c_0, u_0 \text{ constants.} \qquad (79)$$

For this case, from (73) we get a fourth essential coordinate t , and we can conclude that t is a function of u alone, and also that $m(t) = f_0/16$; from (74), by separation, there follows again (76) and (77).
Furthermore, by differentiating (73) with respect to t, and comparing with (76), we obtain once again (78), from which we can get

$$t = \sqrt{2}c_0 \int \frac{\mathrm{d}u}{c_0^2(u - u_0)^2 - f_0}, \qquad (80)$$

where there will be different explicit integrals depending on the nature of f_0; by inversion we obtain $u(t)$.

The functions $s(t), e(t)$ are then determined by making the substitution $u = u(t)$ into the equations (73), (77). As noted above, formally we need to go one order further, but it is easy to confirm that all the Cartan-Karlhede invariants are identically satisfied at fourth order.

Hence Cases **(A)**, **(B)** in Section 4 together correspond directly to Case **(a)** in Sections 2,3.

Case (C) $(ff_{uu} - 5f_u^2/4)f^{-3} = f_0$, constant; $h(u) - g(u)^2/4 = h_0$, constant. These conditions, as we saw in Section 4, correspond to (79) and

$$h(u) = h_0 + g(u)^2/4 \ . \tag{81}$$

For this case, from (73), it follows that $m(t) = f_0/16$, and $s(t) = h_0/2$, and hence from (74) that

$$e(t) = g_u/2\sqrt{2}f^{1/2}. \tag{82}$$

(i) $f^{-1/2}g_u \neq$ constant.
From (82) we get a fourth essential coordinate t, and we can conclude that t is a function of u alone. There is no more information in $\mathcal{R}^3$.
At <u>fourth order</u>, when we equate the invariants $D^4\Phi_{66'}$ and $D^4\tilde{\Phi}_{66'}$, and obtain the major simplification by substituting in all the information from this subcase, we obtain

$$e'(t) = (2f^{-1}g_{uu} - f^{-2}f_u g_u)/8. \tag{83}$$

Then, by differentiating (82) with respect to $t(u)$, and comparing with (83), we obtain again (78) from which we can get again (80). The function $e(t)$ is then determined by making the substitution $u = u(t)$ into the equation (82). Hence we need to go to fourth order of invariants to complete the equivalence problem for this case; furthermore, it is easy to confirm that there is no more information in $\mathcal{R}^4$.

Hence Case **(C(i))** in Section 4 corresponds to Case **(b)** in Sections 3,4.

(ii) $f^{-1/2}g_u = g_0$, constant.
This additional condition, as we saw in Section 4, corresponds to

$$g(u) = c_0 g_0 \int \frac{\mathrm{d}u}{c_0^2(u - u_0)^2 - f_0} \ . \tag{84}$$

In addition, it follows from (82) that $e(t) = g_0/2\sqrt{2}$, and so all three functions $m(t), s(t), e(t)$ are constant in this case.

Hence Case **(C(ii))** in Section 4 corresponds to Case **(c)** in Sections 3,4. (As noted in the earlier sections, the three essential base coordinates are given in $\mathcal{R}^2$ and since there is no further information about essential coordinates supplied at third order, the procedure formally terminates in $\mathcal{R}^3$ for this special case.)

In this case it remains to identify the non-essential coordinate, t. To do this we compare the line elements (19) and (25), by making the substitutions which we

have just identified,

$$c = x, \qquad b = y + g/2, \qquad n = \frac{2fw - xf_u}{(2f)^{3/2}} \tag{85}$$

where $f(u)$ is given by (79) and $g(u)$ by (84). After considerable simplification, it emerges that

$$\frac{\mathrm{d}t}{\mathrm{d}u} = \sqrt{2}f^{1/2} \tag{86}$$

as in the other cases; since $f(u)$ is known in this special case, this yields again (80).

So overall there is equivalence in all cases, with the apparently non-redundant functions (including the special cases when they are non-redundant constants) genuinely non-redundant.

The coordinate freedom for the metric in these coordinates is, in general, given by (85) with the fourth coordinate pair having different transformations depending on the different cases: (78) for Case (**A**), and (80) for Cases (**B**), (**C**), and hence also different relationships, for these different cases, between the respective sets of non-redundant functions.

6 Symmetries

Barnes has, by a long direct calculation [2], integrated the conformal Killing equations for the Edgar-Ludwig metric. From this, he has identified the special cases for a Killing vector, and a homothetic Killing vector. He also identified the very special case involving a two dimensional homothety group; this case had been overlooked in a previous investigation by Edgar and Ludwig in [28]. The coordinates which Barnes uses are very close to the intrinsic coordinates used in Section 2 and 3, and this has meant that his final results are in a comparatively simple form. To carry out an analogous direct integration using the Koutras-McIntosh coordinate version of the metric would be considerably longer and more complicated, and the presentation of the final results would also be in a more complicated form.

However, when we take into account the nature of the coordinates and tetrad in the *intrinsic coordinate version* of the metric (2), it is not necessary to integrate the Killing equations directly, in order to obtain a symmetry analysis. We illustrate, in the next two subsections, how simple such a symmetry analysis for Killing and homothetic Killing vectors can be in intrinsic coordinates, and compare with the much longer and more complicated Killing vector analysis for the Koutras-McIntosh coordinate version in the final subsection.

6.1 Killing vectors in intrinsic coordinate version

An efficient way to investigate Killing vectors and homothetic Killing vectors in the GHP formalism has been developed in [28]. An *intrinsic GHP tetrad* is

a tetrad where the vector directions are fixed by the Riemann tensor and its derivatives; the *intrinsic GHP scalars* are defined, with respect to an intrinsic GHP tetrad, to be the well-behaved GHP spin coefficients, all the Riemann tensor tetrad components, all the GHP derivatives of these spin coefficients and of the Riemann tensor tetrad components, together with properly weighted functional combinations of all of these. It has been shown in [28], that all intrinsic GHP scalars of zero spin and boost weight η, will be Lie derived by any Killing vector $\boldsymbol{\xi}$ present in the spacetime under consideration, i.e., $\xi^i \eta_{,i} = 0$. In the version of the metric (1) found using the GIF and analysed in Section 2, the canonical tetrad used is obviously an intrinsic GHP tetrad; moreover, the GHP Cartan invariants are clearly intrinsic GHP scalars, and since they are directly identified with GHP Cartan invariants, so also are the essential base coordinates a, b, n. Since in addition a, b, n have zero spin and boost weight, they must be Lie derived by any Killing vector $\boldsymbol{\xi}$ present, i.e., $\xi^i a_{,i} = 0$, $\xi^i b_{,i} = 0$, $\xi^i n_{,i} = 0$.

When there is at least one function of t which is not constant, then by a simple coordinate transformation, e.g. $s(t) \to t$, t transforms into a fourth base coordinate which can also be directly identified as a GHP Cartan scalar invariant, and hence $\xi^i t_{,i} = 0$. However, since this implies that all *four* coordinates would have to be Lie derived by any Killing vector, no Killing vector $\boldsymbol{\xi}$ can exist in these circumstances, when at least one of the functions is non-constant.

On the other hand, when all three functions are constant, (respectively m_0, s_0, e_0) then t is a cyclic coordinate not connected to a GHP Cartan invariant, and hence $\xi^i t_{,i} \neq 0$, and so there is one Killing vector, which can be scaled to $\boldsymbol{\xi} = \frac{\partial}{\partial t}$, for the special case of (2) given by

$$
\begin{aligned}
\mathrm{d}s^2 &= \left(a\big(2s_0 - 2am_0 - a^2 - b^2 \big) - e_0^2 - n^2 \right) \mathrm{d}t^2 - 2a\mathrm{d}t\mathrm{d}n + 2n\mathrm{d}t\mathrm{d}a \\
&\quad + 2e_0\mathrm{d}t\mathrm{d}b - \mathrm{d}a^2 - \mathrm{d}b^2 \ .
\end{aligned}
\tag{87}
$$

A fuller discussion of these Killing vector arguments in the GHP formalism, in these coordinates, is given in [12].

6.2 Homothetic Killing vector in intrinsic coordinate version

Koutras and Skea [25] have given an algorithm to determine whether a spacetime admits a homothetic vector[11]. This algorithm was designed to exploit an invariant classification using the Karlhede algorithm, but it is easy to see that it can be used for any invariant classification, as given below:

(1) Use an existing classification algorithm to provide an invariant classification of the spacetime, $\mathcal{R}^n$.

(2) Choose two nonzero elements of $\mathcal{R}^n$ which are *not* both invariant under boosts, and choose a boost which sets the ratio of these elements, raised to their inverse conformal weight, constant.

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[11]Exceptional spacetimes, to which this algorithm is not applicable, are generalised plane waves and homogeneous spacetimes.

(3) In this boosted basis, form the sets of ratios of all nonzero members of $\mathcal{R}^n$ raised to their inverse weights. Call this set $\mathcal{S}^n$.

(4) Calculate the number of functionally independent functions of the coordinates in $\mathcal{S}^n$.

(5) If this number is one less than the number of functions of the coordinates in $\mathcal{R}^n$, the spacetime is homothetic. Otherwise the space-time is not homothetic.

In fact, it is very simple to apply this algorithm in the GHP formalism since it is invariant under spin and boost transformations. Therefore, to illustrate this procedure in these formalisms, we shall use the algorithm on the invariant classification in Section 2.

In this particular case, for simplicity we specialise the spin parameter $p = 1/\sqrt{2}$. Since Φ has conformal weight -2 and boost weight 2, whereas $\partial\Phi$ has conformal weight -3 and boost weight 2, we note that

$$(\Phi)^{-1/2} = (q^2/a)^{-1/2}, \quad \text{and} \quad (\partial\Phi)^{-1/3} = 2^{1/6}(q^2/a^2)^{-1/3}$$

and so to fulfill step (2) of the algorithm we specialise the boost

$$q = a^{-1/2}.$$

The remaining non-trivial GHP Cartan scalars when specialised with this boost are as follows:

First order with conformal weight -3.

$$\text{\thorn}'\Phi = -a^{-3}(na^{-3/2}). \tag{88}$$

Second order with conformal weight -4.

$$\partial\partial\Phi \;=\; a^{-4}, \quad (\partial'\partial\Phi) = \frac{1}{2}a^{-4}, \quad (\partial\text{\thorn}'\Phi) = -a^{-4}\frac{3}{\sqrt{2}}(na^{-3/2}), \tag{89}$$

$$\text{\thorn}'\text{\thorn}'\Phi \;=\; a^{-4}\left(\frac{1}{2} + 3(na^{-3/2})^2 + \frac{1}{2}(ba^{-1})^2 - \left(s(t)a^{-2}\right) + \left(m(t)a^{-1}\right)\right).$$

Third order with conformal weight -5.

$$\partial\partial\text{\thorn}'\Phi \;=\; -6a^{-5}(na^{-3/2}),$$

$$\partial\text{\thorn}'\text{\thorn}'\Phi \;=\; a^{-5}(\frac{1}{\sqrt{2}})\Big(2(ba^{-1})^2 + 15(na^{-3/2})^2 - 4(s(t)a^{-2})$$
$$+3(m(t)a^{-1}) - i(ba^{-1})\Big),$$

$$\text{\thorn}'\text{\thorn}'\text{\thorn}'\Phi \;=\; a^{-5}\Big(-4(na^{-3/2}) - 5(na^{-3/2})(ba^{-1})^2 - 15(na^{-3/2})^3$$
$$-9(na^{-3/2})(m(t)a^{-1}) + 10(na^{-3/2})(s(t)a^{-2}) + (ba^{-1})(e(t)a^{-3/2})$$
$$+(m'(t)a^{-3/2}) - (s'(t)a^{-5/2})\Big). \tag{90}$$

Fourth order with conformal weight -6.

$$\text{\thorn}'X \;=\; a^{-6}\Big((-s''(t)a^{-3}) + (m''(t)a^{-2}) + 4(na^{-3/2})(m'(t)a^{-3/2})$$
$$-5(na^{-3/2})(s'(t)a^{-5/2}) + (ba^{-1})(e'(t)a^{-2})$$
$$-5(ba^{-1})(na^{-3/2})(e(t)a^{-3/2}) + (e(t)a^{-3/2})^2\Big). \tag{91}$$

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The above expressions have been organised so that, when they are raised to their respective inverse conformal weights, their first terms (which will then each be simply a, e.g., for second order invariants $(a^{-4})^{-1/4} = a$) on all the right hand sides will cancel when ratios are taken.[12]

Since the coordinates n, b are rescaled consistently as $(na^{-3/2}), (ba^{-1})$, respectively, then the nature of the ratios will only depend on the nature of the three functions. This means that all the expressions $(s(t)a^{-2}), (m(t)a^{-1}), (e(t)a^{-3/2})$ and $(s'(t)a^{-5/2}), (m'(t)a^{-3/2}), (e'(t)a^{-2})$ and $(s''(t)a^{-3}), (m''(t)a^{-2})$ must each be functions of the same *one* variable, and of zero conformal weight. Of course this cannot happen for the generic metric, but can happen for special cases of the three functions.

For $(s(t)a^{-2}), (s'(t)a^{-5/2}), (s''(t)a^{-3})$ to be functionally dependent on one variable, it is necessary that $s(t) = s_1 t^{-4}$, where s_1 is constant, and similarly it is necessary that $e(t) = e_1 t^{-3}$, and $m(t) = m_1 t^{-2}$, where e_1, m_1 are constants; in all cases we have functions of only the one variable (at^2). Hence the metric (2) will admit a homothetic Killing vector for the special case, when

$$
\begin{aligned}
\mathrm{d}s^2 &= \Big(a\big(2s_1 t^{-4} - 2am_1 t^{-2} - a^2 - b^2\big) - e_1^2 t^{-6} - n^2 \Big) \mathrm{d}t^2 - 2a\mathrm{d}t\mathrm{d}n \\
&\quad + 2n\mathrm{d}t\mathrm{d}a + 2e_1 t^{-3}\mathrm{d}t\mathrm{d}b - \mathrm{d}a^2 - \mathrm{d}b^2
\end{aligned}
\tag{92}
$$

We have already noted that a Killing vector is present when all three functions are constant, so we can make the further deduction of the presence of *both* a Killing and a homothetic Killing vector, i.e., *a two-dimensional homothety*, when all three functions are zero, and

$$
\mathrm{d}s^2 = \Big(-a\big(a^2 + b^2\big) - n^2 \Big)\mathrm{d}t^2 - 2a\mathrm{d}t\mathrm{d}n + 2n\mathrm{d}t\mathrm{d}a - \mathrm{d}a^2 - \mathrm{d}b^2
\tag{93}
$$

In [25] it has also been shown that when there is a homothetic Killing vector $\boldsymbol{\vartheta}$ present, all intrinsic GHP scalars of zero spin and boost weight, η will satisfy the following condition,

$$
\vartheta^i \eta_{,i} = w\sigma\eta
\tag{94}
$$

where σ is the constant homothetic parameter, and w is the conformal weight of η.

As noted in the last subsection, we are using an intrinsic GHP tetrad, and the GHP Cartan invariants are intrinsic GHP scalars. For the subclass of spacetimes with a homothetic Killing vector (92) all four essential base coordinates a, b, n, t are directly identified with GHP Cartan invariants, and since they also have zero spin and boost weight, they must all satisfy condition (94) with respect to their respective conformal weights which are easily deduced from the analysis above; hence the homothetic Killing vector is given by

$$
\boldsymbol{\vartheta} = -\frac{1}{2}t\frac{\partial}{\partial t} + \frac{3}{2}n\frac{\partial}{\partial n} + a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b}
$$

with the parameter choice $\sigma = 1$. These results agree with Barnes in [2].

---

[12]The minor corrections in the numerical coefficients in the GHP scalar invariants compared to the published version [3], do not effect any arguments in this section.

6.3 Killing vectors in Koutras-McIntosh coordinate version.

For comparison, we can investigate the Killing vector in the version of the Edgar-Ludwig spacetime in Koutras-McIntosh coordinates. There are *four* essential base coordinates in all of the cases except Case **(C(ii))**, where there are only three; hence it follows immediately that only in Case **(C(ii))** does a Killing vector exist. The conditions therefore for a Killing vector in this spacetime are the three differential equations for f, g, h respectively

$$(ff_{uu} - 5f_u^2/4)/f^{-3} = f_0,$$

$$h(u) - g(u)^2/4 = h_0$$

$$f^{-1/2}g_u = g_0$$

whose respective integrals are in the text above, (55), (59), (62).

The actual expression for the Killing vector in these coordinates will be quite complicated, and would need to be worked out by directly integrating the Killing equations.

Skea [40], following Koutras [23], investigated the presence of Killing vectors in Koutras-McIntosh coordinates by imposing functional dependence on the four essential base coordinates $\xi_1, \xi_2, \xi_3, \xi_4$ by

$$\frac{\partial(\xi_1, \xi_2, \xi_3, \xi_4)}{\partial(u, w, y, z)} = 0$$

and repeating this for different sets of essential coordinates. This method resulted in a set of three differential equations [40]), each of which is the derivative of one of the equations just quoted. The results, by this method, in [40] agree with the results on Killing vectors quoted above, but the calculations are much longer.

7 Summary and Discussion

This paper highlights the advantages of the version of a spacetime which has been derived in GIF (and so given, as far as possible, in intrinsic coordinates) (2), compared with other more familiar versions. Although the spacetime under consideration here is part of the familiar Kundt family [26], [27] in the Ostváth-Robinson-Rózga form [36] (as recently shown directly by Podolský and Prikryl [38]), we demonstrate how the intrinsic coordinate version gives a different insight into the structure of these spaces.

The simplicity and transparency of this GIF version (2), combined with the fact that we are able to carry it out by hand, gives us a clear unambiguous overview of the invariant classification of this class of metrics. Since the invariant classification procedure is not fully algorithmic, simpler and more transparent calculations give important checks, as well as preventing us overlooking subtle properties. The results obtained in Section 2 have some minor, but subtle and

interesting, disagreements with the conclusions by Edgar and Vickers [16], and by Skea [40]. The full details of the traditional CLASSI analysis of the spacetime in the intrinsic coordinates carried out in Section 3 confirms the results in Section 2.

In addition, using this version of the spacetime (2), we were able to obtain trivially the Killing vector properties, as well as the homothetic Killing vector properties by a straightforward application of the Koutras-Skea algorithm [25]. These simple and efficient calculations are in comparison to the complicated calculations required for the more familiar forms of the spacetime.

The motivation to rework Skea's calculations came from the results of these transparent calculations in Sections 2 and 3. As noted in the Introduction there are subtleties connected with these spacetimes which have revealed short comings in the computer packages [1], [39], which are used to carry out the invariant classification procedure; and once again these spacetimes have revealed subtleties which had not been fully appreciated earlier. In Section 4, the traditional CLASSI analysis of the spacetime in Koutras-McIntosh coordinates is carried out along the same lines as Skea's analysis; the full details of the results of Section 2 are again confirmed, and the discrepencies with Skea's results identified and clarified. Having this analysis alongside the one in intrinsic coordinates demonstrates the simplicity of this GIF version in intrinsic coordinates, and the fewer possibilities for error and misunderstanding.

References

[1] J. Åman. *Manual for CLASSI*, Institute of Theoretical Physics Technical Report, University of Stockholm, 1987.

[2] A. Barnes. *Class. Quantum Grav.*, **18**, 5287 (2001).

[3] M. Bradley, S.B. Edgar and M.P. Machado Ramos. *Gen. Rel. Grav.*, **42**, 155, (2010)

[4] C.H. Brans. *J. Math. Phys.* **6,** 95 (1965).

[5] E. Cartan (1946). *Leçons sur la geometrie des espaces de Riemann* (Gauthier-Villars, Paris).

[6] J Collins, R. A. d'Inverno and J.A. Vickers. *Class. Quantum Grav.*, **7**, 2005 (1991).

[7] J Collins, R. A. d'Inverno and J.A. Vickers. *Class. Quantum Grav.*, **8**, L215 (1991).

[8] J Collins and R. A. d'Inverno. *Class. Quantum Grav.*, **10**, 343 (1993).

[9] S.B Edgar. *Gen Rel Grav.*, **24**, 1267, (1992)

[10] S.B. Edgar and G. Ludwig. *Class. Quantum Grav.*, **14**, L65 (1997).

[11] S.B. Edgar and G. Ludwig. *Gen Rel Grav.*, **29**, 1309 (1997).

[12] S.B. Edgar and G. Ludwig. *Gen. Rel. Grav.*, **32**, 637 (2000).

[13] S.B. Edgar and M.P. Machado Ramos. *Class. Quantum Grav.*, **22**, 791 (2005).

[14] S.B. Edgar and M.P. Machado Ramos. *Gen. Rel. Grav.*, **39**, 539 (2007).

[15] S.B. Edgar and M.P. Machado Ramos. *Gen. Rel. Grav.*, **39**, 1749 (2007).

[16] S.B. Edgar and J.A. Vickers. *Class. Quantum Grav.*, **16**, 589 (1999).

[17] R. Geroch, A. Held and R. Penrose. *J. Math. Phys.*, **14**, 874 (1973).

[18] GRTensorII is a package which runs within Maple. The GRTensorII software and documentation is available freely from http://grtensor.phy.queensu.ca

[19] A. Held. *Gen. Rel. Grav.*, **7**, 177 (1974).

[20] A. Held. *Comm. Math. Phys.*, **44**, 211 (1975).

[21] A. Karlhede. *Gen. Rel. Grav.*, **12**, 693 (1980).

[22] A. Karlhede and M.A.H. MacCallum. *Gen. Rel. Grav.*, **14**, 673 (1982).

[23] A. Koutras. *Class. Quantum Grav.*, **9**, L143 (1992).

[24] A. Koutras and C. McIntosh. *Class. Quantum Grav.*, **13**, L47 (1996).

[25] A. Koutras and J. Skea. *Computer Physics Communications.*, **115**, 350 (1998).

[26] W. Kundt *Z.Phys.*, **163**, 77 (1961).

[27] W. Kundt *Proc. R. Soc. A*, **270**, 328 (1962).

[28] G. Ludwig and S.B. Edgar. *Class. Quantum Grav.*, **17**, 1683 (2000).

[29] G. Ludwig and S.B. Edgar. *Class. Quantum Grav.*, **34**, 807 (2002).

[30] M.A.H. MacCallum and J. E. Åman. *Class. Quantum Grav.*, **3**, 1133 (1986).

[31] M.P. Machado Ramos. *Class. Quantum Grav.*, **15**, 435 (1998).

[32] M.P. Machado Ramos and J.A.G. Vickers. *Proc. Roy. Soc. London A*, **1940**, 693 (1995).

[33] M.P. Machado Ramos and J.A.G. Vickers. *Class. Quantum Grav.*, **13**, 1579 (1996).

[34] M.P. Machado Ramos and J.A.G. Vickers. *Class. Quantum Grav.*, **13**, 1589 (1996).

[35] E. Newman and R. Penrose. *J. Math. Phys.*, **3**, 566 (1962).

[36] I. Ozsváth et al. *J. Math. Phys.*, **26**, 1755 (1985).

[37] G.O Papadopoulos. *J. Math. Phys.*, **47**, 092502-1 (2006).

[38] J. Podolský and O. Prikryl. "On conformally flat and type N pure radiation metrics", *Gen. Rel. Grav.*, **41**, 1069 (2009).

[39] D. Pollney, J.E.F. Skea and R.A. d'Inverno. *Class. Quantum Grav.*, **17**, 2885 (2000).

[40] J.E.F. Skea. *Class. Quantum Grav.*, **14**, 2393 (1997).

[41] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt (2003). *Exact Solutions of Einstein's Equations.* (Cambridge: Cambridge University Press).

[42] P. Wils. *Class. Quantum Grav.*, **6**, 1243 (1989).